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On asymptotic behavior of solutions to linear SDEs with subexponentially stable matrix with applications to stochastic control

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We study the asymptotic behavior of solutions to linear stochastic differential equations with subexponentially stable matrix. The result established in the form of the strong law of large numbers is then applied to linear-quadratic Gaussian control problems over an infinite time-horizon for undetectable systems.

Let $X_t, t \geq 0$, be an n-dimensional stochastic process defined on a stochastic basic by

$$dX_t = A_t X_t \, dt + G_t dw_t \,, \quad X_0 = x \,,$$

where A_t, G_t are known non-random matrices, $\int\limits_0^\infty \|G_t\|^2 \, dt > 0$, ($\|\cdot\|$ – Euclidean matrix norm); $w_t, t \geq 0$, – is a d-dimensional Brownian motion; x is a non-random vector.

The problem of interest is to investigate the asymptotic behavior of $\|X_t\|^2$ in the sense of the strong law of large numbers [1]. More specifically, we are to consider the ratio

$$Z_t = \|X_t\|^2 / \int\limits_0^t \|G_s\|^2 \, ds,$$

as $t \to \infty$. Here the non-random denominator represents the total variance of cumulative disturbances. It is known (see [2]) that $Z_t \to 0$ a.s. if A_t is exponentially stable. We will study the case when $\|A_t\| \to 0, t \to \infty$, i.e. A_t does not possess such kind of stability. First of all, a weaker notion of stability should be introduced.

Definition 1. We say that A_t is subexponentially stable with the rate $\delta_t > 0$ if
i) $\delta_t \to 0, t \to \infty$ and $\|A_t\| \leq \kappa \delta_t$, $t \geq 0$, where κ is a constant;
ii)

$$\|\Phi(t,s)\| \leq \kappa_1 e^{-\kappa_2 \int\limits_s^t \delta_v \, dv} \,, \quad s \leq t \,,$$

for some constants $\kappa_1, \kappa_2 > 0$; $\Phi(t,s)$ is the fundamental matrix corresponding to A_t;
iii) $\int\limits_0^t \delta_s \, ds \to \infty$, $t \to \infty$.

One important implication of exponentially stable A_t is the bounded $E\|X_t\|^2$. For the case of subexponential stability we require the following

Assumption $\mathcal{AG}$. Let A_t and G_t be such that
a) A_t is subexponentially stable with the rate δ_t;
b) $E\|X_t\|^2$ is bounded, $t \geq 0$.

Remark. b) is satisfied if $\|G_t\|^2 \leq \bar{c}\delta_t$, $t \geq 0$, for some constant $\bar{c} > 0$.

The main result is stated below.

Theorem 1. Assume $\mathcal{AG}$. Then

$$\lim_{t\to\infty} \frac{\|X_t\|^2}{\int\limits_0^t \|G_s\|^2\, ds} = 0\,, \quad \text{a.s.},\ t \to \infty\,.$$

Next we discuss some applications of the established result to LQG control problems for undetectable systems over an infinite time-horizon.

The controlled stochastic process $Y_t, t \geq 0$, is governed by

$$dY_t = C_t Y_t dt + B_t U_t dt + V_t dw_t\,, \qquad Y_0 = y, \tag{1}$$

where $C_t,\ B_t,\ V_t$ are bounded, $\|C_t\| \to 0, t \to \infty$; $U_t,\ t \geq 0$, is an admissible control, i.e. an $\mathcal{F}_t = \sigma\{w_s,\ s \leq t\}$–adapted k-dimensional process such that there exists a solution to (1). Let us denote by $\mathcal{U}$ the set of admissible controls. The cost functional over the planning horizon $[0, T]$ is given by

$$J_T(U) = \int_0^T [Y_t' Q_t Y_t + U_t' U_t]\, dt,$$

where Q_t is symmetric and positive semidefinite, $\|Q_t\| \to 0, t \to \infty$.

Since we have asymptotically singular C_t and Q_t, the pair $(A_t, \sqrt{Q_t})$ is undetectable. In the absence of detectability, the well-known feedback control law $U_t^* = -B_t'\Pi_t Y_t^*$, if it exists, might not provide the exponentially stable $C_t - B_t B_t' \Pi_t$ in the equation

$$dY_t^* = (C_t - B_t B_t' \Pi_t) Y_t^* dt + V_t dw_t\,, \quad Y_0^* = y\,,$$

where the bounded absolute continuous function $\Pi_t,\ t \geq 0$, with values in the set of symmetric positive semidefinite matrices, satisfies the Riccati equation

$$\dot{\Pi}_t + \Pi_t C_t + C_t' \Pi_t - \Pi_t B_t B_t' \Pi_t + Q_t = 0\,.$$

As a result, U^* could not be optimal as $T \to \infty$ with respect to the long-run (pathwise) average or (pathwise) extended long-run average cost criteria. However, subexponential stability assumption on $C_t - B_t B_t' \Pi_t + (1/2)\dot{p}_t/p_t$ along with $\||\Pi_t\| \leq p_t \to 0,\ t \to \infty$, included into a set of conditions guarantee average (or pathwise) optimality of U^*. The optimality can be obtained with the use of Theorem 1 for $X_t = \sqrt{p_t} Y_t^*$.

Источники и литература

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